

Structure of wrap groups of quaternion and octonion fiber bundles.

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Abstract

This article is devoted to the investigation of structure of wrap groups of connected fiber bundles over the fields of real \mathbf{R} , complex \mathbf{C} numbers, the quaternion skew field \mathbf{H} and the octonion algebra \mathbf{O} . Iterated wrap groups are studied as well. Their smashed products are constructed.

1 Introduction.

Geometric loop groups of circles were first introduced by Lefschetz in 1930-th and then their construction was reconsidered by Milnor in 1950-th. Lefschetz has used the C^0 -uniformity on families of continuous mappings, which led to the necessity of combining his construction with the structure of a free group with the help of words. Later on Milnor has used the Sobolev's H^1 -uniformity, that permitted to introduce group structure more naturally [26].

The construction of Lefschetz is very restrictive, because it works with the C^0 uniformity of continuous mappings in compact-open topology. Even for spheres S^n of dimension $n > 1$ it does not work directly, but uses the iterated loop group construction of circles. Then their constructions were generalized for fibers over circles and spheres with parallel transport structures over \mathbf{C} . Smooth Deligne cohomologies were studied on such groups [5].

Wrap groups of quaternion and octonion fibers as well as for wider classes of fibers over \mathbf{R} or \mathbf{C} were defined and various examples were given together with basic theorems in [12]. In that paper a construction of wrap groups was done with the help of Sobolev uniformities, that has permitted to consider wide families of manifolds and fiber bundles. This paper continues previous works of the author on this theme [13, 21, 19, 20, 12]. Wrap groups are

generalizations of geometric loop groups from spheres onto a wider class of manifolds and fiber bundles over them.

Geometric loop groups have important applications in modern physical theories (see [9, 23] and references therein). Groups of loops are also intensively used in gauge theory. Wrap groups can be used in the membrane theory which is the generalization of the string (superstring) theory.

In paper [12] wrap groups of fiber bundles over quaternions and octonions were defined and investigated and numerous examples were described. This article is devoted to investigation of their structure and uses notations and results of the previous work. Besides quaternion and octonion also real and complex fiber bundles are considered with wrap groups for them. Smashed products of wrap groups are constructed. Iterated wrap groups are studied as well.

All main results of this paper are obtained for the first time and they are given in Theorems 2, 6, 9, 10, 20, 21, Propositions 3, 7, 8, 12, 13, 17 and Corollary 11.

Remind the basic definitions and notations.

1. Note. Denote by \mathcal{A}_r the Cayley-Dickson algebra such that $\mathcal{A}_0 = \mathbf{R}$, $\mathcal{A}_1 = \mathbf{C}$, $\mathcal{A}_2 = \mathbf{H}$ is the quaternion skew field, $\mathcal{A}_3 = \mathbf{O}$ is the octonion algebra. Henceforth we consider only $0 \leq r \leq 3$.

2.1. Remark. If M is a metrizable space and $K = K_M$ is a closed subset in M of codimension $\text{codim}_{\mathbf{R}} N \geq 2$ such that $M \setminus K = M_1$ is a manifold with corners over \mathcal{A}_r , then we call M a pseudo-manifold over \mathcal{A}_r , where K_M is a critical subset.

Two pseudo-manifolds B and C are called diffeomorphic, if $B \setminus K_B$ is diffeomorphic with $C \setminus K_C$ as for manifolds with corners (see also [5, 24]).

Take on M a Borel σ -additive measure ν such that ν on $M \setminus K$ coincides with the Riemann volume element and $\nu(K) = 0$, since the real shadow of M_1 has it.

The uniform space $H_p^t(M_1, N)$ of all continuous piecewise H^t Sobolev mappings from M_1 into N is introduced in the standard way [19, 20], which induces $H_p^t(M, N)$ the uniform space of continuous piecewise H^t Sobolev mappings on M , since $\nu(K) = 0$, where $\mathbf{R} \ni t \geq [m/2] + 1$, m denotes the dimension of M over \mathbf{R} , $[k]$ denotes the integer part of $k \in \mathbf{R}$, $[k] \leq k$. Then put $H_p^\infty(M, N) = \bigcap_{t > m} H_p^t(M, N)$ with the corresponding uniformity.

For manifolds over \mathcal{A}_r with $1 \leq r \leq 3$ take as $H_p^t(M, N)$ the completion of the family of all continuous piecewise \mathcal{A}_r -holomorphic mappings from M into N relative to the H_p^t uniformity, where $[m/2] + 1 \leq t \leq \infty$. Henceforth we consider pseudo-manifolds with connecting mappings of charts continuous in M and $H_p^{t'}$ in $M \setminus K_M$ for $0 \leq r \leq 3$, where $t' \geq t$.

2.2. Note. Since the octonion algebra \mathbf{O} is non-associative, we consider

a non-associative subgroup G of the family $Mat_q(\mathbf{O})$ of all square $q \times q$ matrices with entries in \mathbf{O} . More generally G is a group which has a H_p^t manifold structure over \mathcal{A}_r and group's operations are H_p^t mappings. The G may be non-associative for $r = 3$, but G is supposed to be alternative, that is, $(aa)b = a(ab)$ and $a(a^{-1}b) = b$ for each $a, b \in G$.

As a generalization of pseudo-manifolds there is used the following (over \mathbf{R} and \mathbf{C} see [5, 31]). Suppose that M is a Hausdorff topological space of covering dimension $\dim M = m$ supplied with a family $\{h : U \rightarrow M\}$ of the so called plots h which are continuous maps satisfying conditions (D1 – D4):

- (D1) each plot has as a domain a convex subset U in \mathcal{A}_r^n , $n \in \mathbf{N}$;
- (D2) if $h : U \rightarrow M$ is a plot, V is a convex subset in \mathcal{A}_r^t and $g : V \rightarrow U$ is an H_p^t mapping, then $h \circ g$ is also a plot, where $t \geq [m/2] + 1$;
- (D3) every constant map from a convex set U in \mathcal{A}_r^n into M is a plot;
- (D4) if U is a convex set in \mathcal{A}_r^n and $\{U_j : j \in J\}$ is a covering of U by convex sets in \mathcal{A}_r^n , each U_j is open in U , $h : U \rightarrow M$ is such that each its restriction $h|_{U_j}$ is a plot, then h is a plot. Then M is called an H_p^t -differentiable space.

A mapping $f : M \rightarrow N$ between two H_p^t -differentiable spaces is called differentiable if it is continuous and for each plot $h : U \rightarrow M$ the composition $f \circ h : U \rightarrow N$ is a plot of N . A topological group G is called an H_p^t -differentiable group if its group operations are H_p^t -differentiable mappings.

Let E, N, F be $H_p^{t'}$ -pseudo-manifolds or $H_p^{t'}$ -differentiable spaces over \mathcal{A}_r , let also G be an $H_p^{t'}$ group over \mathcal{A}_r , $t \leq t' \leq \infty$. A fiber bundle $E(N, F, G, \pi, \Psi)$ with a fiber space E , a base space N , a typical fiber F and a structural group G over \mathcal{A}_r , a projection $\pi : E \rightarrow N$ and an atlas Ψ is defined in the standard way [5, 24, 33] with the condition, that transition functions are of $H_p^{t'}$ class such that for $r = 3$ a structure group may be non-associative, but alternative.

Local trivializations $\phi_j \circ \pi \circ \Psi_k^{-1} : V_k(E) \rightarrow V_j(N)$ induce the $H_p^{t'}$ -uniformity in the family W of all principal $H_p^{t'}$ -fiber bundles $E(N, G, \pi, \Psi)$, where $V_k(E) = \Psi_k(U_k(E)) \subset X^2(G)$, $V_j(N) = \phi_j(U_j(N)) \subset X(N)$, where $X(G)$ and $X(N)$ are \mathcal{A}_r -vector spaces on which G and N are modelled, $(U_k(E), \Psi_k)$ and $(U_j(N), \phi_j)$ are charts of atlases of E and N , $\Psi_k = \Psi_k^E$, $\phi_j = \phi_j^N$.

If $G = F$ and G acts on itself by left shifts, then a fiber bundle is called the principal fiber bundle and is denoted by $E(N, G, \pi, \Psi)$. As a particular case there may be $G = \mathcal{A}_r^*$, where \mathcal{A}_r^* denotes the multiplicative group $\mathcal{A}_r \setminus \{0\}$. If $G = F = \{e\}$, then E reduces to N .

3. Definitions. Let M be a connected H_p^t -pseudo-manifold over \mathcal{A}_r , $0 \leq r \leq 3$ satisfying the following conditions:

(i) it is compact;

(ii) M is a union of two closed subsets over \mathcal{A}_r A_1 and A_2 , which are pseudo-manifolds and which are canonical closed subsets in M with $A_1 \cap A_2 = \partial A_1 \cap \partial A_2 =: A_3$ and a codimension over \mathbf{R} of A_3 in M is $\text{codim}_{\mathbf{R}} A_3 = 1$, also A_3 is a pseudo-manifold;

(iii) a finite set of marked points $s_{0,1}, \dots, s_{0,k}$ is in $\partial A_1 \cap \partial A_2$, moreover, ∂A_j are arcwise connected $j = 1, 2$;

(iv) $A_1 \setminus \partial A_1$ and $A_2 \setminus \partial A_2$ are H_p^t -diffeomorphic with $M \setminus [\{s_{0,1}, \dots, s_{0,k}\} \cup (A_3 \setminus \text{Int}(\partial A_1 \cap \partial A_2))]$ by mappings $F_j(z)$, where $j = 1$ or $j = 2$, $\infty \geq t \geq [m/2] + 1$, $m = \dim_{\mathbf{R}} M$ such that $H^t \subset C^0$ due to the Sobolev embedding theorem [25], where the interior $\text{Int}(\partial A_1 \cap \partial A_2)$ is taken in $\partial A_1 \cup \partial A_2$.

Instead of (iv) we consider also the case

(iv') M , A_1 and A_2 are such that $(A_j \setminus \partial A_j) \cup \{s_{0,1}, \dots, s_{0,k}\}$ are $C^0([0, 1], H_p^t(A_j, A_j))$ -retractable on $X_{0,q} \cap A_j$, where $X_{0,q}$ is a closed arcwise connected subset in M , $j = 1$ or $j = 2$, $s_{0,q} \in X_{0,q}$, $X_{0,q} \subset K_M$, $q = 1, \dots, k$, $\text{codim}_{\mathbf{R}} K_M \geq 2$.

Let \hat{M} be a compact connected H_p^t -pseudo-manifold which is a canonical closed subset in \mathcal{A}_r^l with a boundary $\partial \hat{M}$ and marked points $\{\hat{s}_{0,q} \in \partial \hat{M} : q = 1, \dots, 2k\}$ and an H_p^t -mapping $\Xi : \hat{M} \rightarrow M$ such that

(v) Ξ is surjective and bijective from $\hat{M} \setminus \partial \hat{M}$ onto $M \setminus \Xi(\partial \hat{M})$ open in M , $\Xi(\hat{s}_{0,q}) = \Xi(\hat{s}_{0,k+q}) = s_{0,q}$ for each $q = 1, \dots, k$, also $\partial M \subset \Xi(\partial \hat{M})$.

A parallel transport structure on a $H_p^{t'}$ -differentiable principal G -bundle $E(N, G, \pi, \Psi)$ with arcwise connected E and G for H_p^t -pseudo-manifolds M and \hat{M} as above over the same \mathcal{A}_r with $t' \geq t + 1$ assigns to each H_p^t mapping γ from M into N and points $u_1, \dots, u_k \in E_{y_0}$, where y_0 is a marked point in N , $y_0 = \gamma(s_{0,q})$, $q = 1, \dots, k$, a unique H_p^t mapping $\mathbf{P}_{\hat{\gamma}, u} : \hat{M} \rightarrow E$ satisfying conditions (P1 – P5):

(P1) take $\hat{\gamma} : \hat{M} \rightarrow N$ such that $\hat{\gamma} = \gamma \circ \Xi$, then $\mathbf{P}_{\hat{\gamma}, u}(\hat{s}_{0,q}) = u_q$ for each $q = 1, \dots, k$ and $\pi \circ \mathbf{P}_{\hat{\gamma}, u} = \hat{\gamma}$

(P2) $\mathbf{P}_{\hat{\gamma}, u}$ is the H_p^t -mapping by γ and u ;

(P3) for each $x \in \hat{M}$ and every $\phi \in \text{Diff}_p^t(\hat{M}, \{\hat{s}_{0,1}, \dots, \hat{s}_{0,2k}\})$ there is the equality $\mathbf{P}_{\hat{\gamma}, u}(\phi(x)) = \mathbf{P}_{\hat{\gamma} \circ \phi, u}(x)$, where $\text{Diff}_p^t(\hat{M}, \{\hat{s}_{0,1}, \dots, \hat{s}_{0,2k}\})$ denotes the group of all H_p^t homeomorphisms of \hat{M} preserving marked points $\phi(\hat{s}_{0,q}) = \hat{s}_{0,q}$ for each $q = 1, \dots, 2k$;

(P4) $\mathbf{P}_{\hat{\gamma}, u}$ is G -equivariant, which means that $\mathbf{P}_{\hat{\gamma}, uz}(x) = \mathbf{P}_{\hat{\gamma}, u}(x)z$ for every $x \in \hat{M}$ and each $z \in G$;

(P5) if U is an open neighborhood of $\hat{s}_{0,q}$ in \hat{M} and $\hat{\gamma}_0, \hat{\gamma}_1 : U \rightarrow N$ are $H_p^{t'}$ -mappings such that $\hat{\gamma}_0(\hat{s}_{0,q}) = \hat{\gamma}_1(\hat{s}_{0,q}) = v_q$ and tangent spaces, which are vector manifolds over \mathcal{A}_r , for γ_0 and γ_1 at v_q are the same, then the

tangent spaces of $\mathbf{P}_{\hat{\gamma}_0, u}$ and $\mathbf{P}_{\hat{\gamma}_1, u}$ at u_q are the same, where $q = 1, \dots, k$, $u = (u_1, \dots, u_k)$.

Two $H_p^{t'}$ -differentiable principal G -bundles E_1 and E_2 with parallel transport structures (E_1, \mathbf{P}_1) and (E_2, \mathbf{P}_2) are called isomorphic, if there exists an isomorphism $h : E_1 \rightarrow E_2$ such that $\mathbf{P}_{2, \hat{\gamma}, u}(x) = h(\mathbf{P}_{1, \hat{\gamma}, h^{-1}(u)}(x))$ for each H_p^t -mapping $\gamma : M \rightarrow N$ and $u_q \in (E_2)_{y_0}$, where $q = 1, \dots, k$, $h^{-1}(u) = (h^{-1}(u_1), \dots, h^{-1}(u_k))$.

Let $(S^M E)_{t, H} := (S^{M, \{s_{0, q} : q=1, \dots, k\}} E; N, G, \mathbf{P})_{t, H}$ be a set of H_p^t -closures of isomorphism classes of H_p^t principal G fiber bundles with parallel transport structure.

2 Structure of wrap groups

1. Proposition. *The H_p^m uniformity in $L(S^m, N)$ (see §2.10 in [12]) for $m > 1$ is strictly stronger, than the m times iterated H_p^1 uniformity.*

Proof. If $f \in H^m$, then $\partial^k f(x) / \partial x_1^{k_1} \dots \partial x_m^{k_m} \in L^2$ for each $0 \leq k \leq m$, $k = k_1 + \dots + k_m$, $0 \leq k_j, j = 1, \dots, m$. But g of m times iterated H^1 uniformity means that $\partial^k g(x) / \partial x_1^{k_1} \dots \partial x_m^{k_m} \in L^2$ for each $0 \leq k \leq m$, $k = k_1 + \dots + k_m$, $0 \leq k_j \leq 1, j = 1, \dots, m$. The latter conditions are weaker than that of H^m . For $m > 1$ there may appear g for which such partial derivatives are not in L^2 , when $1 < k_j \leq m$. Using transition mappings of charts of atlases $At(M)$ and $At(N)$ and applying this locally we get the statement.

2. Theorem. *For a wrap group $W = (W^M E)_{t, H}$ (see Definition 2.7 [12]) there exists a skew product $\hat{W} = W \tilde{\otimes} W$ which is an H_p^l alternative Lie group and there exists a group embedding of W into \hat{W} , where $l = t' - t$ ($l = \infty$ for $t' = \infty$), $E = E(N, G, \pi, \Psi)$ is a principal G -bundle of class $H_p^{t'}$ with $t' \geq t \geq [\dim(M)/2] + 1$. If G is associative, then \hat{W} is associative. Moreover, the loop group $L(S^1, E)$ is H_p^t isomorphic with $(\hat{W}^{S^1} E)_{t, H}$ in the particular case of S^1 .*

Proof. Let \tilde{W} be a set of all elements $(g_1 a_1 \otimes g_2 a_2) \in (W \otimes B)^2$, where B is a free non-commutative associative group with two generators a, b , $ab \neq ba$, $g_1, g_2 \in W$. Take in \tilde{W} the equivalence relation: $g_1 g_2 a \otimes g_2 b \stackrel{\sim}{=} g_1 e_B \otimes e e_B$, for each $g_1, g_2 \in W$, where e and e_B denote the unit elements in W and in B .

Define in \tilde{W} the multiplication:

$$(g_1 a_1 \otimes g_2 a_2) \tilde{\otimes} (g_3 a_3 \otimes g_4 a_4) := ((g_1 g_3)(a_1 a_3) \otimes (g_4 g_2)((a_1^{-1} a_4 a_1) a_2))$$

for each $g_1, g_2, g_3, g_4 \in W$ and every $a_1, a_2, a_3, a_4 \in B$, hence

$$\begin{aligned} (e \otimes g_1 a_1) \tilde{\otimes} (e \otimes g_2 a_2) &= e \otimes (g_2 g_1)(a_2 a_1), \\ (g_1 a_1 \otimes e) \tilde{\otimes} (g_2 a_2 \otimes e) &= (g_1 g_2)(a_1 a_2) \otimes e, \\ (g_1 a_1 \otimes e) \tilde{\otimes} (e \otimes g_4 a_4) &= g_1 a_1 \otimes g_4(a_1^{-1} a_4 a_1), \end{aligned}$$

$$(e \otimes g_4 a_4) \tilde{\otimes} (g_1 a_1 \otimes e) := g_1 a_1 \otimes g_4 a_4.$$

Thus this semidirect product \tilde{W} of groups $(W \otimes B) \otimes^s (W \otimes B)$ is non-commutative, since $b^{-1}aba^{-1} \neq e$, where $e := e \times e_B$, \otimes^s denotes the semidirect product, \otimes denotes the direct product.

Consider the minimal closed subgroup A in the semidirect product \tilde{W} generated by elements $(g_1 g_2 a \otimes g_2 b) \tilde{\otimes} (g_1 e_B \otimes e e_B)^{-1}$, where B is supplied with the discrete topology and \tilde{W} is supplied with the product uniformity. Then put $\hat{W} := \tilde{W}/A =: W \tilde{\otimes} W$ and denote the multiplication in \hat{W} as in \tilde{W} .

Therefore, W has the group embedding $\theta : g \mapsto (g e_B \otimes e)$ into \hat{W} and the multiplication $m[(g_1 e_B \otimes e), (g_2 e_B \otimes e)] = (g_1 e_B \otimes e) \tilde{\otimes} (g_2 e_B \otimes e)$.

On the other hand, $(g a_1 \otimes e) \tilde{\otimes} (e \otimes g a_1 a_2 a_1^{-1}) = g a_1 \otimes g a_2 = (e \otimes e) =: \tilde{e}$, $\tilde{e} = \tilde{e} A = A$ is the unit element in \hat{W} and $(e \otimes g a_1 a_2 a_1^{-1}) = (g a_1 \otimes e)^{-1}$ is the inverse element of $(g a_1 \otimes e)$, where $a_2 \in B$ is such that $(a_1 \otimes a_2) \tilde{\otimes} A = (e \otimes e) \tilde{\otimes} A = A$ in \hat{W} , $a_1 = e a_1$, that is $a_1 \otimes a_2 \tilde{=} e \otimes e$ in \tilde{W} .

From preceding formulas it follows, that \hat{W} is noncommutative and alternative. As the manifold \hat{W} is the quotient of the H_p^t manifold W^2 by the H_p^t equivalence relation, hence \hat{W} is the H_p^t differentiable space, since Conditions (D1–D4) of §2.1.3.2 [12] are satisfied. The group operation and the inversion in \hat{W} combines the product in W and the inversion with the tensor product and the equivalence relation, hence they are H_p^l differentiable with $l = t' - t$, $l = \infty$ for $t' = \infty$, (see §§1.11, 1.12, 1.15 in [31] and §2.1.3.1 in [12]).

Then $((g_1 \otimes g_2) \tilde{\otimes} (g_3 \otimes g_4)) \tilde{\otimes} (g_5 \otimes g_6) := ((g_1 g_3) g_5 \otimes g_6 (g_4 g_2))$ and

$$(g_1 \otimes g_2) \tilde{\otimes} ((g_3 \otimes g_4) \tilde{\otimes} (g_5 \otimes g_6)) := (g_1 (g_3 g_5) \otimes (g_6 g_4) g_2).$$

Therefore, \hat{W} is alternative, since W is alternative (see Theorem 2.6.1 [12]) and B is associative. If G is associative, then W is associative and \hat{W} is associative.

Consider the commutator

$$\begin{aligned} & [(g_1 a_1 \otimes g_2 a_2) \tilde{\otimes} (g_3 a_3 \otimes g_4 a_4)] \tilde{\otimes} [(g_1 a_1 \otimes g_2 a_2)^{-1} \tilde{\otimes} \\ & (g_3 a_3 \otimes g_4 a_4)^{-1}] = \{((g_1 g_3)(a_1 a_3) \otimes (g_4 g_2)((a_1^{-1} a_4 a_1) a_2)) \tilde{\otimes} \\ & [(g_1^{-1} a_1^{-1} \otimes g_2^{-1}(a_1 a_2^{-1} a_1^{-1})) \tilde{\otimes} (g_3^{-1} a_3^{-1} \otimes g_4^{-1}(a_3 a_4^{-1} a_3^{-1}))]\} \\ & = ((g_1 g_3)(a_1 a_3) \otimes (g_4 g_2)((a_1^{-1} a_4 a_1) a_2)) \tilde{\otimes} ((g_1^{-1} g_3^{-1})(a_1^{-1} a_3^{-1}) \otimes (g_4^{-1} g_2^{-1}) \\ & (a_1 (a_3 a_4^{-1} a_3^{-1}) a_1^{-1})(a_1 a_2^{-1} a_1^{-1}))) = (((g_1 g_3)(g_1^{-1} g_3^{-1}))(a_1 a_3 a_1^{-1} a_3^{-1}) \otimes \\ & ((g_4^{-1} g_2^{-1})(g_4 g_2))((a_1 a_3)^{-1} [((a_1 a_3) a_4^{-1} (a_1 a_3)^{-1})(a_1 a_2^{-1} a_1^{-1})] (a_1 a_3)) ((a_1^{-1} a_4 a_1) a_2)). \end{aligned}$$

The minimal closed subgroup generated by products of such elements is the commutant \tilde{W}_c of \tilde{W} . The group $(W^M N)_{t,H}$ is commutative (see Theorem 6(2) [12]). We have $B/B_c = \{e\}$, the quotient group $G/G_c = G_{ab}$ is the abelianization of G , particularly if G is commutative, then $G_{ab} = G$, where G_c denotes the commutant subgroup of G . Therefore,

$(W^M E; N, G, \mathbf{P})_{t,H} / [(W^M E; N, G, \mathbf{P})_{t,H}]_c = (W^M E; N, G_{ab}, \mathbf{P})_{t,H}$
and inevitably $\tilde{W}/\tilde{W}_c = (W^M E; N, G_{ab}, \mathbf{P})_{t,H}$.

Using the equivalence relation in \tilde{W} we get $\hat{W}/\hat{W}_c = (W^M E; N, G_{ab}, \mathbf{P})_{t,H}$.

In the particular case of $M = S^1$ for $g \in W$ take $f \in g$, that is $\langle f \rangle_{t,H} = g$. The equivalence class of f relative to the analogous closures of orbits of the right action of the subgroup $Diff_+^\infty(S^1, s_0)$ preserving a marked point and an orientation of S^1 induced by that of $I = [0, 1]$ denote by $[f]_{t,H}$, then to $[f]_{t,H}$ put into the correspondence $ga \otimes e$ in \tilde{W} , while to $[f^-]_{t,H}$ counterpose $e \otimes gaba^{-1}$, where $f^-(x) := f(1-x)$ for each $x \in [0, 1]$, the unit circle S^1 is parametrized as $z = e^{2\pi ix}$, $z \in S^1 \subset \mathbf{C}$, $x \in [0, 1]$. Their equivalence classes $(ga \otimes e) \tilde{\otimes} A$ and $(e \otimes gaba^{-1}) \tilde{\otimes} A$ in \tilde{W} give elements in \hat{W} .

Since $[f]_{t,H}^{-1} := [f^-]_{t,H}$ and $[f_1 \vee f_2]_{t,H} = [f_1]_{t,H} [f_2]_{t,H}$, then \hat{W} is isomorphic with $L(S^1, E)_{t,H}$.

3. Proposition. *If there exists an $H_p^{t'}$ -diffeomorphism $\eta : N \rightarrow N$ such that $\eta(y_0) = y_0'$, where $t \leq t'$ then wrap groups $(W^M E; y_0)_{t,H}$ and $(W^M E; y_0')_{t,H}$ defined with marked points y_0 and y_0' are H_p^l -isomorphic as H_p^l -differentiable groups, where $l = t' - t$ for finite t' , $l = \infty$ for $t' = \infty$.*

Proof. Let $f \in H_p^t(M, E)$, then $\eta \circ \pi \circ f(s_{0,q}) = \eta(y_0) = y_0'$ for each marked point $s_{0,q}$ in M , where $\pi : E \rightarrow N$ is the projection, $\pi \circ f = \gamma$, γ is a wrap, that is an H_p^t -mapping from M into N with $\gamma(s_{0,q}) = y_0$ for $q = 1, \dots, k$. The manifold N is connected together with E and G in accordance with conditions imposed in [12]. Consider the $H_p^{t'}$ -diffeomorphism $\eta \times e$ of the principal bundle E . Then $\Theta : H_p^t(M, W) \rightarrow H_p^t(M, W)$ is the induced isomorphism such that $\pi \circ \Theta(f) := \eta \circ \pi \circ f : M \rightarrow N$ and $(\eta \times e) \circ f = \Theta(f)$ for $f \in H_p^t(M, E)$. The mapping Θ is H_p^l differentiable by f , hence it gives the H_p^l isomorphism of the considered H_p^l -differentiable wrap groups (see Theorem 6(1) [12]).

4. Remark. As usually we suppose, that the principal bundle E , its structure group G and the base manifold N are arcwise connected. Let $(\mathcal{P}^M E)_{t,H}$ be a space of equivalence classes $\langle f \rangle_{t,H}$ of $f \in H_p^t(M, W)$ relative to the closures of orbits of the left action of $Diff H_p^t(M; \{s_{0,q} : q = 1, \dots, k\})$. This means, that $(\mathcal{P}^M E)_{t,H}$ is the quotient space of $H_p^t(M, W)$ relative to the equivalence relation $R_{t,H}$.

There is the embedding $\theta : H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; W) \hookrightarrow H_p^t(M; W)$ and the evaluation mapping $\hat{e}v : H_p^t(M; W) \rightarrow N^k$ such that $\hat{e}v(f) := (\hat{f}(\hat{s}_{0,q}) : q = k+1, \dots, 2k)$, $\hat{e}v_{\hat{s}_{0,q}}(f) := \hat{f}(\hat{s}_{0,q})$, where $\hat{f} \in H_p^t(\hat{M}; W)$ is such that $\hat{f} = f \circ \Xi$, $\Xi : \hat{M} \rightarrow M$ is the quotient mapping. We get the diagram $H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; W) \rightarrow H_p^t(M; W) \rightarrow N^k$ with H_p^t differentiable mappings, which induces the diagram $H_p^{t,l+1}(M, \{s_{0,q} : q = 1, \dots, k\}; W, y_0) \rightarrow H_p^t(M, H_p^{t,l}(M, \{s_{0,q} : q = 1, \dots, k\}; W, y_0) \rightarrow H_p^{t,l}(M, \{s_{0,q} :$

$q = 1, \dots, k\}; W, y_0)$ for each $l \in \mathbf{N}$, where $H_p^{t,l+1}(M, \{s_{0,q} : q = 1, \dots, k\}; W, y_0) := H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; H_p^{t,l}(M, \{s_{0,q} : q = 1, \dots, k\}; W, y_0))$, $H_p^{t,1}(M, \{s_{0,q} : q = 1, \dots, k\}; W, y_0) := H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; W, y_0)$. Therefore, there exist iterated wrap semigroups and groups $(S^M E)_{l+1;t,H} := (S^M(S^M E)_{l;t,H})_{t,H}$ and $(W^M E)_{l+1;t,H} := (W^M(W^M E)_{l;t,H})_{t,H}$, where $(S^M E)_{1;t,H} := (S^M E)_{t,H}$ and $(W^M E)_{1;t,H} := (W^M E)_{t,H}$.

Evidently, if there are H_p^t and $H_p^{t'}$ diffeomorphisms $\rho : M \rightarrow M_1$ and $\eta : N \rightarrow N_1$ mapping marked points into respective marked points, then $H_p^t(M, W)$ is isomorphic with $H_p^t(M_1, W_1)$ and hence $(W^M E)_{b;t,H}$ is H_p^t isomorphic as the H_p^t -manifold and H_p^l -isomorphic as the H_p^l -Lie group with $(W^{M_1} E_1)_{b;t,H}$ for each $b \in \mathbf{N}$, where $l = t' - t$, $l = \infty$ for $t' = \infty$, $t' \geq t \geq [\dim(M)/2] + 1$. If $f : N \rightarrow N_1$ is a surjective map and N is an H_p^t -differentiable space, then N inherits a structure of an H_p^t -differentiable space with plots having the local form $f \circ \rho : U \rightarrow N_1$, where $\rho : U \rightarrow N$ is a plot of N .

5. Lemma. *Let E be an $H_p^{t'}$ principal bundle and let D be an everywhere dense subset in N such that for each $y \in D$ there exists an open neighborhood V of y in N and a differentiable map $p : V \rightarrow H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; V, y) := \{f \in H_p^t(M; V) : f(s_{0,q}) = y, q = 1, \dots, k\}$ such that $\hat{e}v_{\hat{s}_{0,q}}(\hat{p}(y)) = y$ for each $q = 1, \dots, 2k$ and each $y \in N$, where $p \circ \Xi = \pi \circ \hat{p}$. Then $\hat{e}v : H_p^t(M; W) \rightarrow N^k$ is an H_p^t differentiable principal $(S^M E)_{t,H}$ bundle.*

Proof. Let $\{(V_j, y_j) : j \in J\}$ be a family such that $y_j \in V_j \cap D$ for each j and there exists $p_j : V_j \rightarrow H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; V_j, y_j)$ so that $\hat{p}_j(\hat{s}_{0,q})(y) = y \times e$ for each $q = 1, \dots, 2k$ and every j , where $\{V_j : j \in J\}$ is an open covering of N , y is a constant mapping from \hat{M} into V_j with $y(\hat{M}) = \{y\}$, where $\hat{p}_j(\hat{s}_{0,q})$ is the restriction to V_j of the projection $\hat{p}(\hat{s}_{0,q}) : (\mathcal{P}^M E)_{t,H} \rightarrow E$, while $p_j(\Xi(\hat{x}))(y) = \pi \circ \hat{p}_j(\hat{x})(y \times e)$ for each $y \in N$ and $x = \Xi(\hat{x})$ in M , where $\hat{x} \in \hat{M}$, $\Xi : \hat{M} \rightarrow M$. Then $(W^M E)_{t,H}$ and $(\mathcal{P}^M E)_{t,H}$ are supplied with the H_p^t -differentiable spaces structure (see Remark 4 above and Theorem 6 [12]), where the embedding $(S^M E)_{t,H} \hookrightarrow (\mathcal{P}^M E)_{t,H}$ and the projection $\hat{e}v_{\hat{s}_{0,q}} : (\mathcal{P}^M E)_{t,H} \rightarrow N$ are H_p^t -maps.

Let $\psi_j \in \text{Diff}_p^t(N)$ such that $\psi_j(y) = y_j$. Specify a trivialization $\phi_j : \hat{p}_j^{-1}(\hat{s}_{0,q})(V_j) \rightarrow V_j \times (S^M E)_{t,H}$ of the restriction $\hat{p}_j(\hat{s}_{0,q})|_{V_j}$ of the projection $\hat{p}_j(\hat{s}_{0,q}) : (\mathcal{P}^M E)_{t,H} \rightarrow E$ by the formula $\phi_j(f) = (f(\hat{s}_{0,q}), \psi_j \circ \hat{p}_j(\hat{s}_{0,q})(f))$ for each $f \in (\mathcal{P}^M E)_{t,H}$ with $\pi \circ f(\hat{s}_{0,q}) = y$, where $\psi_j \circ \hat{p}_j(f) = \psi_j(\hat{p}_j(f))$. Then $\phi_j^{-1}(y, g) = g^{-1}(\psi_j \circ \hat{p}_j(y)) =: \eta$, $\eta \in (\mathcal{P}^M E)_{t,H}$ with $\pi \circ \psi_j \circ f(\hat{s}_{0,q}) = y_j$, since G is a group, where $g = \psi_j \circ \hat{p}_j(f)$. Finally the combination of the family $\{\hat{e}v_{\hat{s}_{0,q}} : q = k+1, \dots, 2k\}$ induce the mapping $\hat{e}v : H_p^t(M; W) \rightarrow N^k$. By the construction a fiber of this bundle is the monoid $(S^M E)_{t,H}$.

6. Theorem. *If N is a smooth manifold over \mathcal{A}_r (holomorphic for $1 \leq r \leq 3$ respectively), then there exists an H_p^t -differentiable principal $(S^M E)_{t,H}$ bundle $\hat{e}v : (\mathcal{P}^M E)_{t,H} \rightarrow N^k$.*

Proof. In view of Lemma 5 it is sufficient to prove that for each $y \in N$ there exists a neighborhood U of y in N and an H_p^t -map $p_q : U \rightarrow H_p^t(M, W)$ such that $ev_{s_{0,q}}(p_q(z)) = z$ for each $q = 1, \dots, k$, $z \in U$, where $ev_x(f) = f(x)$.

In \hat{M} consider a rectifiable curve $\zeta_q : [0, 1] \rightarrow \hat{M}$ joining $\hat{s}_{0,q}$ with $\hat{s}_{0,q+k}$, where $1 \leq q \leq k$. Then consider a coordinate system (x_1, \dots, x_m) in \hat{M} such that x_1 corresponds to a natural coordinate along ζ_q . This coordinate system is defined locally for each chart of \hat{M} and x_1 is defined globally.

Consider a real shadow $N_{\mathbf{R}}$ of N , then $N_{\mathbf{R}}$ is the Riemann C^∞ manifold. Thus there exists a Riemannian metric \mathbf{g} in N . For each $y \in N$ there exists a geodesic ball U at y of radius less than the injectivity radius \exp^N for \mathbf{g} . Then there exists a map $p_q : U \rightarrow (\mathcal{P}^M U)_{t,H}$ with $\pi \circ [p_q(\hat{s}_{0,q+k})(z)] = z$ and $\pi \circ [p_q(\hat{s}_{0,q})(z)] = y$ for each $z \in U$, where $p_q \circ \zeta_q =: \hat{\gamma}_{q,y,z}$ is the shortest geodesic in U joining y with z , $\hat{\gamma}_{q,y,z} : [0, 1] \rightarrow N$, $\hat{\gamma}_{q,y,z} \circ \zeta_q^{-1}(x_1) \in N$ for each x_1 . Having initially $\hat{\gamma}_{q,y,z}$ extend it to \hat{p}_q on \hat{M} with values in E such that $p_q \circ \Xi = \pi \circ \hat{p}_q$.

7. Proposition. (1). *The wrap group $(W^M E; N, G, \mathbf{P})_{t,H}$ is the principal G^k bundle over $(W^M N)_{t,H}$.*

(2). *The abelianization $[(W^M E; N, G, \mathbf{P})_{t,H}]_{ab}$ of the wrap group $(W^M E; N, G, \mathbf{P})_{t,H}$ is isomorphic with $(W^M E; N, G_{ab}, \mathbf{P})_{t,H}$.*

(3). *For $n \geq 2$ the iterated loop group $(L^{S^n} E)_{t,H}$ is isomorphic with the wrap group $(W^{S^n} E)_{t,H}$ for the sphere S^n and a principal fiber bundle E for $\dim_{\mathbf{R}} N \geq 2$ with $k = 1$.*

Proof. 1. The bundle structure $\pi : E \rightarrow N$ induces the bundle structure $\hat{\pi} : (W^M E; N, G, \mathbf{P})_{t,H} \rightarrow (W^M N)_{t,H}$, since $\pi \circ \mathbf{P}_{\hat{\gamma},u} = \hat{\gamma}$. In view of Lemma 5 it is sufficient to show, that there exists a neighborhood U_G of e in $(W^M E)_{t,H}$ and a G -equivariant mapping $\phi : U_G \rightarrow (W^M N)_{t,H}$. Let $\langle \mathbf{P}_{\hat{\gamma},u} \rangle_{t,H} \in (W^M E)_{t,H}$, where $\hat{\gamma} : \hat{M} \rightarrow N$, $\hat{\gamma} = \gamma \circ \Xi$, $\gamma : M \rightarrow N$, $\gamma(s_{0,q}) = y_0$ for each $q = 1, \dots, k$. Then $\pi \circ \mathbf{P}_{\hat{\gamma},u} = \hat{\gamma}$ and $\mathbf{P}_{\hat{\gamma},u}$ is G -equivariant by the conditions defining the parallel transport structure, that is $\mathbf{P}_{\hat{\gamma},u}(x)z = \mathbf{P}_{\hat{\gamma},uz}(x)$ for each $x \in \hat{M}$ and $z \in G$ and every $u \in E_{y_0}$. We have that $uG = \pi^{-1}(y)$ for each $u \in E_y$ and $y \in N$.

Therefore, put $\phi = \pi_*$, where $\pi_* \langle \mathbf{P}_{\hat{\gamma},u} \rangle_{t,H} = \langle \hat{\gamma}, u \rangle_{t,H}$ and take $U_G = \pi_*^{-1}(U)$, where U is a symmetric $U^{-1} = U$ neighborhood of e in $(W^M N)_{t,H}$.

The group G acts effectively on E . Since G is arcwise connected, then G^k acts effectively on $(W^M E)_{t,H}$. Indeed, for each ζ_q from §6 there is $g_q \in G$ corresponding to $\hat{\gamma}(\hat{s}_{0,q+k})$ with $\mathbf{P}_{\hat{p}_q, \hat{s}_{0,q} \times e}(\hat{s}_{0,q+k}) = \{y_0 \times g_q\} \in E_{y_0}$, $g_q \in G$ for

every $q = 1, \dots, k$. Moreover, $\pi_*^{-1}(\pi_*(\langle \mathbf{P}_{\hat{\gamma},u} \rangle_{t,H})) = \langle \mathbf{P}_{\hat{\gamma},u} \rangle_{t,H} G^k$. Then the fibre of $\hat{\pi} : (W^M E; N, G, \mathbf{P})_{t,H} \rightarrow (W^M N)_{t,H}$ is G^k . Due to Conditions 2(P1 – P5) [12] it is the principal G^k differentiable bundle of class H_p^t .

2, 3. In view of Proposition 1 the loop group $(L^{S^n} E)_{l,H}$ is everywhere dense in the n times iterated loop group $(L^{S^1}(\dots(L^{S^1} E)_{1,H}\dots))_{1,H}$, while the wrap group $(W^{S^n} E)_{l,H}$ is everywhere dense in the n times iterated wrap group $(W^{S^1} E)_{n;1,H}$ for each $l \geq n$. For each $n > m$ there exists the natural projection $\pi_n^m : S^n \rightarrow S^m$ which induces the embeddings $(W^{S^m} E)_{t,H} \hookrightarrow (W^{S^n} E)_{t,H}$ and $(L^{S^m} E)_{t,H} \hookrightarrow (L^{S^n} E)_{t,H}$ in accordance with Corollary 9 [12], since $k = 1$ and choosing a marked point $s_0 \in S^1$. Therefore, due to $\dim_{\mathbf{R}} N \geq 2$ the considered here wrap and loop groups are infinite dimensional. Therefore, statements (2, 3) follow from (1) and the proof of Theorem 2 above and Proposition 11 [12] in accordance with which the iterated loop group $(L^{S^1}(\dots(L^{S^1} E)_{1,H}\dots))_{1,H}$ is commutative.

8. Proposition. *If E is contractible, then $(\mathcal{P}^M E)_{t,H}$ is contractible.*

Proof. Let $g : [0, 1] \times E \rightarrow E$ be a contraction such that g is continuous and $g(0, z) = z$ and $g(1, z) = y_0 \times e$ for each $z \in E$. Then for each $f \in H_p^t(M, W)$ we get $g(0, f(x)) = f(x)$ and $g(1, f(x)) = y_0 \times e$ for each $x \in M$. Moreover, $g(s, \langle f \rangle_{t,H}) \subset \langle g(s, f) \rangle_{t,H}$ for each $s \in [0, 1]$, since $f \in g_s^{-1}(\langle g(s, f) \rangle_{t,H})$ and g is continuous while $\langle g(s, f) \rangle_{t,H}$ by its definition is closed in $H_p^t(M, W)$, where $g_s(z) := g(s, z)$. Therefore, $id = g(0, *) : (\mathcal{P}^M E)_{t,H} \rightarrow (\mathcal{P}^M E)_{t,H}$ and $g(1, (\mathcal{P}^M E)_{t,H}) = \langle w_0 \rangle_{t,H}$.

8.1. Notation. Denote by $Hom_p^t((W^M E)_{t,H}, G)$ or $Hom_p^t((S^M E)_{t,H}, G)$ the group or the monoid of H_p^t differentiable homomorphisms from $(W^M E)_{t,H}$ or $(S^M E)_{t,H}$ respectively into G . By \mathcal{A}_r^* is denoted the multiplicative group of $\mathcal{A}_r \setminus \{0\}$, where $0 \leq r \leq 3$.

9. Theorem. *Let $Dif H_p^{t'}(N)$ acts transitively on N , $t \leq t'$. For each H^∞ manifold N and an H_p^t differentiable group G such that $\mathcal{A}_r^* \subset G$ with $1 \leq r \leq 3$ there exists a homomorphism of the H_p^t differentiable space of all equivalence classes of $(\mathcal{P}^M E)_{t,H}$ relative to $Dif H_p^{t'}(N)$ (see §§1.3.2 and 3 [12]) and $Hom_p^t((S^M E)_{t,H}, G^k)$. They are isomorphic, when G is commutative.*

Proof. Mention that due to Theorem 6 the H_p^t -differentiable principal $(S^M E)_{t,H}$ bundle $\hat{e}v : (\mathcal{P}^M E)_{t,H} \rightarrow N^k$ has a parallel transport structure $\hat{\mathbf{P}}_{\hat{\gamma},uz}(x) = \hat{\mathbf{P}}_{\hat{\gamma},u}(x)z$ for each $x \in \hat{M}$ and all $\gamma \in H_p^t(M, N)$ and $u \in \hat{e}v^{-1}(\gamma(s_{0,k}))$ and every $z \in G$ and the corresponding $\hat{\gamma} : \hat{M} \rightarrow N$ such that $\gamma \circ \Xi = \hat{\gamma}$. If $x = \hat{s}_{0,q}$ with $1 \leq q \leq k$, then $\hat{\mathbf{P}}$ gives the identity homomorphism from $(S^M E)_{t,H}$ into $(S^M E)_{t,H}$. If $\theta : (S^M E)_{t,H} \rightarrow G^k$ is an H_p^t differentiable homomorphism, then the holonomy of the associated parallel transport $\hat{\mathbf{P}}^\theta$ on the bundle $(\mathcal{P}^M E)_{t,H} \times^\theta G \rightarrow N^k$ is the homomor-

phism $\theta : (S^M E)_{t,H} \rightarrow G^k$ (see §2.3 in [12]). At the same time the group G contains continuous one-parameter subgroups from \mathcal{A}_r^* , where $1 \leq r \leq 3$. If $g \in (W^M N)_{t,H}$ and $g \neq e$, then g is of infinite order, since w_0 does not belong to g^n for each $n \neq 0$ non-zero integer n , where $w_0(M) = \{y_0\}$.

This holonomy induces a map $h : (\mathcal{P}^M E)_{t,H} / \mathcal{Q} \rightarrow \text{Hom}_p^t((S^M E)_{t,H}, G^k)$, where \mathcal{Q} is an equivalence relation caused by the transitive action of $\text{Dif} H_p^{t'}(N)$ such that $(S^M E)_{t,H}$ with distinct marked points either $\{s_{0,q} : q = 1, \dots, k\}$ in M and y_0 or \tilde{y}_0 in N are isomorphic, since there exists $\psi \in \text{Dif} H_p^{t'}(N)$ such that $\psi(y_0) = \tilde{y}_0$.

If G is commutative, then this map is the homomorphism, since $(S^M E)_{t,H}$ is the commutative monoid for a commutative group G (see Theorem 3.2 [12]) and $u\mathbf{P}_{\hat{\gamma}_1, v_1}(x_1)\mathbf{P}_{\hat{\gamma}_2, v_2}(x_2) = u\mathbf{P}_{\hat{\gamma}_2, v_2}(x_2)\mathbf{P}_{\hat{\gamma}_1, v_1}(x_1)$ for each $x_1, x_2 \in \hat{M}$ and $u, v_1, v_2 \in E_{y_0}$. There is the embedding $(S^M E)_{t,H} \hookrightarrow (W^M E)_{t,H}$, hence a homomorphism $\theta : (W^M E)_{t,H} \rightarrow G^k$ has the restriction on $(S^M E)_{t,H}$ which is also the homomorphism.

For $G \supset \mathcal{A}_r^*$ there exists a family of $f \in \text{Hom}_p^t((S^M E)_{t,H}, G^k)$ separating elements of the wrap monoid $(S^M E)_{t,H}$, hence there exists the embedding of $(S^M E)_{t,H}$ into $\text{Hom}_p^t((S^M E)_{t,H}, G^k)$. The bundle $(\mathcal{P}^M E)_{t,H} \times^\theta G \rightarrow N^k$ has the induced parallel transport structure \mathbf{P}^θ . The holonomy of the parallel transport structure on $(\mathcal{P}^M N)_{t,H} \times^\theta G \rightarrow N^k$ is θ . Therefore, the map $H_p^t((S^M E)_{t,H}, G^k) \ni \theta \mapsto \mathbf{P}^\theta$ is inverse to h .

10. Theorems. *Suppose that $M_2 \hookrightarrow M_1$ and $M = M_1 \setminus (M_2 \setminus \partial M_2)$ and $\hat{M}_2 \hookrightarrow \hat{M}_1$ and $\hat{M} = \hat{M}_1 \setminus (\hat{M}_2 \setminus \partial \hat{M}_2)$ and $N_2 \hookrightarrow N_1$ are H_p^t -pseudo-manifolds with the same marked points $\{s_{0,q} : q = 1, \dots, k\}$ for M_1 and M_2 and M and $y_0 \in N_2$ satisfying conditions of §2 [12] and G_2 is a closed subgroup in G_1 with a topologically complete principal fiber bundle E with a structure group G_1 .*

1. *Then $(W^{M_2, \{s_{0,q}:q=1,\dots,k\}} E; N_2, G_2, \mathbf{P})_{t,H}$ has an embedding as a closed subgroup into $(W^{M_1, \{s_{0,q}:q=1,\dots,k\}} E; N_1, G_1, \mathbf{P})_{t,H}$.*

2. *The wrap group $(W^{M_2, \{s_{0,q}:q=1,\dots,k\}} E; N, G_2, \mathbf{P})_{t,H}$ is normal in $(W^{M_1, \{s_{0,q}:q=1,\dots,k\}} E; N, G_1, \mathbf{P})_{t,H}$ if and only if G_2 is a normal subgroup in G_1 .*

3. *In the latter case $(W^M E; N, G, \mathbf{P})_{t,H}$ is isomorphic with $(W^{M_1} E; N, G_1, \mathbf{P})_{t,H} / (W^{M_2} E; N, G_2, \mathbf{P})_{t,H}$, where $G = G_1 / G_2$.*

Proof. 1. If $\hat{\gamma}_2 \in H_p^t(\hat{M}_2, N_2)$, then it has an H_p^t extension to $\hat{\gamma}_1 \in H_p^t(\hat{M}_1, N_1)$ due to Theorem III.4.1 [25]. Therefore, the parallel transport structure $\mathbf{P}_{\hat{\gamma}_1, u}$ over \hat{M}_1 serves as an extension of $\mathbf{P}_{\hat{\gamma}_2, u}$ over \hat{M}_2 . The uniform spaces $H_p^t(M_j, \{s_{0,1}, \dots, s_{0,k}\}; W_j, y_0)$ are complete for $j = 1, 2$, since the principal fiber bundle E is topologically complete and the corresponding principal fiber sub-bundle E_2 with the structure group G_2 is also com-

plete (see Theorem 8.3.6 [4]). Therefore, $H_p^t(M_2, \{s_{0,1}, \dots, s_{0,k}\}; W_2, y_0)$ has embedding as the closed subspace into $H_p^t(M_1, \{s_{0,1}, \dots, s_{0,k}\}; W_1, y_0)$. Each H_p^t diffeomorphism of M_2 has an H_p^t extension to a diffeomorphism of M_1 (see also §III.4 in [25] and [35]). Since G_2 is a closed subgroup in G_1 , then $(S^{M_2, \{s_{0,q}:q=1,\dots,k\}}E; N_2, G_2, \mathbf{P})_{t,H}$ has an embedding as a closed sub-monoid into $(S^{M_1, \{s_{0,q}:q=1,\dots,k\}}E; N_1, G_1, \mathbf{P})_{t,H}$ and inevitably $(W^{M_2, \{s_{0,q}:q=1,\dots,k\}}E; N_2, G_2, \mathbf{P})_{t,H}$ has an embedding as a closed subgroup into $(W^{M_1, \{s_{0,q}:q=1,\dots,k\}}E; N_1, G_1, \mathbf{P})_{t,H}$ due to Theorem 6.1 [12].

2. The groups $(W^{M_j, \{s_{0,q}:q=1,\dots,k\}}N)_{t,H}$ for $j = 1, 2$ are commutative and $(W^{M_j, \{s_{0,q}:q=1,\dots,k\}}E)_{t,H}$ is the G_j^k principal fiber bundle on $(W^{M_j, \{s_{0,q}:q=1,\dots,k\}}N)_{t,H}$ (see Theorem 6.2 [12] and Proposition 7.1 above). Therefore, $(W^{M_2, \{s_{0,q}:q=1,\dots,k\}}E)_{t,H}$ is the normal subgroup in $(W^{M_1, \{s_{0,q}:q=1,\dots,k\}}E)_{t,H}$ if and only if G_2 is the normal subgroup in G_1 .

3. Consider the principal fiber bundle $E(N, G, \pi, \Psi)$ with the structure group G (see Note 1.3.2 [12]) and the parallel transport structure \mathbf{P} for the H_p^t pseudo-manifold \hat{M} , where $G = G_1/G_2$ is the quotient group. If $\hat{\gamma}_1 \in H_p^t(\hat{M}_1, N)$, then $\hat{\gamma}_1$ is the combination

$$(i) \quad \hat{\gamma}_1 = \hat{\gamma}_2 \nabla \hat{\gamma},$$

where $\hat{\gamma}_2$ and $\hat{\gamma}$ are restrictions of $\hat{\gamma}_1$ on \hat{M}_2 and \hat{M} correspondingly. On the other hand, each $\hat{\gamma} \in H_p^t(\hat{M}, N)$ has an extension $\hat{\gamma}_1 \in H_p^t(\hat{M}_1, N)$. The manifold \hat{M}_1 is metrizable by a metric ρ . For each $\epsilon > 0$ there exists $\psi \in \text{Diff}_p^t(\hat{M}_1; \{\hat{s}_{0,q} : q = 1, \dots, 2k\})$ such that $(\psi(\hat{M}) \cap \hat{M}_2) \subset \bigcup_{l=1}^s B(\hat{M}_1, x_l, \epsilon)$ for some $x_l \in \hat{M}_1$ with $l = 1, \dots, s$ and $s \in \mathbf{N}$ and $\psi|_{\hat{M}_1 \setminus (\hat{M} \bigcup_{l=1}^s B(\hat{M}_1, x_l, \epsilon))} = id$, since \hat{M}_1 and \hat{M}_2 are compact pseudo-manifolds. Therefore, using Lemma 2.1.3.16 [20] and charts of the manifolds gives

$$< \mathbf{P}_{\hat{\gamma}, u}|_M >_{t,H} = < \mathbf{P}_{\hat{\gamma}_1, u}|_{M_1} >_{t,H} / < \mathbf{P}_{\hat{\gamma}_2, u}|_{M_2} >_{t,H}$$

due to decomposition (i), since $\mathbf{P}_{\hat{\gamma}, u}|_{M_j} \in G_j$ for $j = 1, 2$ and $G = G_1/G_2$ is the $H_p^{t'}$ quotient group with $t' \geq t$. Consequently, $(W^M E; N, G, \mathbf{P})_{t,H}$ is isomorphic with

$$(W^{M_1} E; N, G_1, \mathbf{P})_{t,H} / (W^{M_2} E; N, G_2, \mathbf{P})_{t,H} \text{ (see also §§3, 6 [12]).}$$

11. Corollary. *Let suppositions of Theorem 10 be satisfied. Then $(W^M N)_{t,H}$ is isomorphic with $(W^{M_1} N)_{t,H} / (W^{M_2} N)_{t,H}$.*

Proof. For $(W^M N)_{t,H}$ taking $G = G_1 = G_2 = \{e\}$ we get the statement of this corollary from Theorem 10.3.

12. Proposition. *Suppose that $M = M_1 \vee M_2$, where M_1 and M_2 are H_p^t -pseudo-manifolds satisfying Conditions 2.2(i-v) [12] with the bunch taken by marked points $\{s_{0,q} : q = 1, \dots, k\}$, then $(W^M N)_{t,H}$ is isomorphic with the internal direct product $(W^{M_1} N)_{t,H} \otimes (W^{M_2} N)_{t,H}$.*

Proof. The manifold M has marked points $\{s_{0,q} : q = 1, \dots, k\}$ such

that $s_{0,q}$ corresponds to $s_{0,q,1}$ glued with $s_{0,q,2}$ in the bunch $M_1 \vee M_2$ for each $q = 1, \dots, k$, where $s_{0,q,j} \in M_j$ are marked points $j = 1, 2$. Since each M_j satisfies Conditions 2.2($i-v$) [12], then M satisfies them also.

In view of Theorem 10.1 $(W^{M_j, \{s_{0,q}: q=1, \dots, k\}} N)_{t,H}$ has an embedding as a closed subgroup into $(W^{M, \{s_{0,q}: q=1, \dots, k\}} N)_{t,H}$ for $j = 1, 2$. If $\gamma_j \in H_p^t(M_j, \{s_{0,q} : q = 1, \dots, k\}; N, y_0)$ for $j = 1, 2$, then $\gamma_1 \vee \gamma_2 \in H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; N, y_0)$. On the other hand, each $\gamma \in H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; N, y_0)$ has the decomposition $\gamma = \gamma_1 \vee \gamma_2$, where $\gamma_j = \gamma|_{M_j}$ for $j = 1, 2$. Therefore, $\langle \gamma \rangle_{t,H} = \langle \gamma_1 \vee \gamma_2 \rangle_{t,H} = \langle \gamma_1 \rangle_{t,H} \vee \langle \gamma_2 \rangle_{t,H}$, where $w_0(M) = \{y_0\}$, $w_{0,j} = w_0|_{M_j}$ for $j = 1, 2$, hence $(W^M N)_{t,H}$ is isomorphic with $(W^{M_1} N)_{t,H} \otimes (W^{M_2} N)_{t,H}$.

13. Propositions. 1. *Let $\theta : N_1 \rightarrow N$ be an embedding with $\theta(y_1) = y_0$, or $F : E_1 \rightarrow E$ be an embedding of principal fiber bundles over \mathcal{A}_r such that $\pi \circ F|_{N_1 \times e} = \theta \circ \pi_1$, then there exist embeddings $\theta_* : (W^M N_1)_{t,H} \rightarrow (W^M N)_{t,H}$ and $F_* : (W^M E_1)_{t,H} \rightarrow (W^M E)_{t,H}$.*

2. *If $\theta : N_1 \rightarrow N$ and $F : E_1 \rightarrow E$ are a quotient mapping and a quotient homomorphism such that N_1 is a covering pseudo-manifold of a pseudo-manifold N , then $(W^M N)_{t,H}$ is the quotient group of some closed subgroup in $(W^M N_1)_{t,H}$ and $(W^M E)_{t,H}$ is the quotient group of some closed subgroup in $(W^M E_1)_{t,H}$.*

3. *If there are an H_p^t diffeomorphism $f_1 : M \rightarrow M_1$ and an $H_p^{t'}$ -isomorphism $f_2 : E \rightarrow E_1$, then wrap groups $(W^{M_1} E_1)_{t,H}$ and $(W^M E)_{t,H}$ are isomorphic.*

Proof. 1. If $\gamma_1 \in H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; N_1, y_1)$, then $\theta \circ \gamma_1 = \gamma \in H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; N, y_0)$, $\langle \gamma \rangle_{t,H} = \theta_* \langle \gamma_1 \rangle_{t,H}$, where $\theta_* \langle \gamma_1 \rangle_{t,H} := \{\theta \circ f : f R_{t,H} \gamma_1\}$. In addition $F|_{E_1, v}$ gives an embedding $F : G_1 \rightarrow G$, where G_1 and G are structural groups of E_1 and E . Therefore, for the parallel transport structures we get

$$(1) F \circ \mathbf{P}_{\hat{\gamma}_1, v}^1(x) = \mathbf{P}_{\hat{\gamma}, u}(x)$$

for each $x \in \hat{M}$, where $F(v) = u$, $\pi \circ F = \theta \circ \pi_1$, where \mathbf{P}^1 is for E_1 and \mathbf{P} for E . Define $F_* \langle \mathbf{P}_{\hat{\gamma}_1, v}^1 \rangle_{t,H} := \{F \circ g : g R_{t,H} \mathbf{P}_{\hat{\gamma}_1, v}^1\}$. Since θ and F are H_p^t differentiable mappings, then θ_* and F_* are embeddings of H_p^t manifolds and group homomorphisms of H_p^l differentiable groups (see also Theorems 6 [12]).

2. If $\gamma \in H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; N, y_0)$, then there exists $\gamma_1 \in H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; N_1, y_1)$ such that $\theta \circ \gamma_1 = \gamma$, since N_1 is a covering of N , that is each $y \in N$ has a neighborhood V_y for which $\theta^{-1}(V_y)$ is a disjoint union of open subsets in N_1 for each $y \in N$. This γ_1 exists due to connectedness of M and $\gamma(M)$, where $\gamma(M) \subset N$. To each parallel transport in E_1 there corresponds a parallel transport in E so that Equation (1) above is satisfied. Put $\theta_*^{-1} \langle \gamma \rangle_{t,H} = \{\langle \gamma_1 \rangle_{t,H} : \theta \circ \gamma_1 = \gamma\}$ and

$F_*^{-1} < \mathbf{P}_{\hat{\gamma},u} >_{t,H} := \{ < \mathbf{P}_{\hat{\gamma}_1,v}^1 >_{t,H} : F \circ \mathbf{P}_{\hat{\gamma}_1,v}^1 = \mathbf{P}_{\hat{\gamma},u} \}$, where $F(v) = u$.

This gives quotient mappings θ_* and F_* from closed subgroups $\theta_*^{-1}(W^M N)_{t,H}$ and $F_*^{-1}(W^M E)_{t,H}$ in $(W^M N_1)_{t,H}$ and $(W^M E_1)_{t,H}$ respectively onto $(W^M N)_{t,H}$ and $(W^M E)_{t,H}$ by closed subgroups $\theta_*^{-1}(e)$ and $F_*^{-1}(e)$ correspondingly.

3. We have that $g \in H_p^t(M, \{s_{0,q} : q = 1, \dots, k\}; W, y_0)$ if and only if $f_2 \circ g \circ f_1^{-1} \in H_p^t(M_1, \{s_{0,q,1} : q = 1, \dots, k\}; W_1, y_1)$, where $f_1(s_{0,q}) = s_{0,q,1}$ for each $q = 1, \dots, k$, $f_2(y_0 \times e) = y_1 \times e$. At the same time $\psi \in \text{Dif} H_p^t(M)$ if and only if $f_1 \circ \psi \circ f_1^{-1} \in \text{Dif} H_p^t(M_1)$. Hence $(S^M E)_{t,H}$ is isomorphic with $(S^{M_1} E_1)_{t,H}$ and inevitably wrap groups $(W^M E)_{t,H}$ and $(W^{M_1} E_1)_{t,H}$ are H_p^t diffeomorphic as manifolds and isomorphic as H_p^t groups.

14. Note. If N is a manifold not necessarily orientable, then it contains up to equivalence of atlases a connected chart V open in N such that $y \in V$ and V is orientable. Since $(W^M E|_V)_{t,H}$ is the infinite dimensional group, then $(W^M E)_{t,H}$ is also infinite dimensional even if N is not orientable due to Proposition 13.1. If N is not orientable, then there exists an orientable covering manifold N_1 and a quotient mapping $\theta : N_1 \rightarrow N$ as in Proposition 13(2) (see also about coverings and orientable coverings in §§50, 51 [29], §§II.4.18,19 [2]).

It is necessary to mention that some circumstances of wrap groups are related also with their infinite dimensionality.

15. Note. Let G be a topological group not necessarily associative, but alternative:

(A1) $g(gf) = (gg)f$ and $(fg)g = f(gg)$ and $g^{-1}(gf) = f$ and $(fg)g^{-1} = f$ for each $f, g \in G$

and having a conjugation operation which is a continuous automorphism of G such that

(C1) $\text{conj}(gf) = \text{conj}(f)\text{conj}(g)$ for each $g, f \in G$,

(C2) $\text{conj}(e) = e$ for the unit element e in G .

If G is of definite class of smoothness, for example, H_p^t differentiable, then conj is supposed to be of the same class. For commutative group in particular it can be taken the identity mapping as the conjugation. For $G = \mathcal{A}_r^*$ it can be taken $\text{conj}(z) = \tilde{z}$ the usual conjugation for each $z \in \mathcal{A}_r^*$, where $1 \leq r \leq 3$.

Suppose that

(A2) $\hat{G} = \hat{G}_0 i_0 \oplus \hat{G}_1 i_1 \oplus \dots \oplus \hat{G}_{2r-1} i_{2r-1}$ such that G is a multiplicative group of a ring \hat{G} with the multiplicative group structure, where $\hat{G}_0, \dots, \hat{G}_{2r-1}$ are pairwise isomorphic commutative associative rings and $\{i_0, \dots, i_{2r-1}\}$ are generators of the Cayley-Dickson algebra \mathcal{A}_r , $1 \leq r \leq 3$ and $(y_l i_l)(y_s i_s) = (y_l y_s)(i_l i_s)$ is the natural multiplication of any pure states in G for $y_l \in G_l$. For example, $G = (\mathcal{A}_r^*)^n$ and $\hat{G} = \mathcal{A}_r^n$.

16. Lemma. If G and K are two topological or differentiable groups

twisted over $\{i_0, \dots, i_{2^r-1}\}$ satisfying conditions 15(A1, A2, C1, C2) and K is a closed normal subgroup in G , where $2 \leq r \leq 3$, then the quotient group is topological or differentiable and twisted over $\{i_0, \dots, i_{2^r-1}\}$.

Proof. Since $\hat{G} = \hat{G}_0 i_0 \oplus \hat{G}_1 i_1 \oplus \dots \oplus \hat{G}_{2^r-1} i_{2^r-1}$, where $\hat{G}_0, \dots, \hat{G}_{2^r-1}$ are pairwise isomorphic, then $\hat{G}/\hat{K} = (\hat{G}_0/\hat{K}_0) i_0 \oplus \dots \oplus (\hat{G}_{2^r-1}/\hat{K}_{2^r-1}) i_{2^r-1}$ is also twisted. Each \hat{G}_j is associative, hence G/K is alternative, since $2 \leq r \leq 3$ and using multiplicative properties of generators of the Cayley-Dickson algebra \mathcal{A}_r . On the other hand, $\text{conj}(K) = K$, hence $\text{conj}(gK) = K \text{conj}(g) = \text{conj}(g)K \in G/K$ and $\text{conj}(ghK) = \text{conj}(gh)K = (\text{conj}(h)\text{conj}(g))K = (\text{conj}(h)K)(\text{conj}(g)K) = \text{conj}(hK)\text{conj}(gK) = \text{conj}(gKhK)$.

The subgroup K is closed in G , hence by the definition of the quotient differentiable structure G/K is the differentiable group (see also §§1.11, 1.12, 1.15 in [31]).

17. Proposition. Let $\eta : N_1 \rightarrow N_2$ be an $H_p^{t'}$ -retraction of $H_p^{t'}$ manifolds, $N_2 \subset N_1$, $\eta|_{N_2} = \text{id}$, $y_0 \in N_2$, where $t' \geq t$, M is an H_p^t manifold, $E(N_1, G, \pi, \Psi)$ and $E(N_2, G, \pi, \Psi)$ are principal $H_p^{t'}$ bundles with a structure group G satisfying conditions of §2 [12]. Then η induces the group homomorphism η_* from $(W^M E; N_1, G, \mathbf{P})_{t,H}$ onto $(W^M E; N_2, G, \mathbf{P})_{t,H}$.

Proof. In view of Proposition 7(1) the wrap group $(W^M E; N_1, G, \mathbf{P})_{t,H}$ is the principal G^k bundle over $(W^M N_1)_{t,H}$. Extend η to $\vartheta : E(N_1, G, \pi, \Psi) \rightarrow E(N_2, G, \pi, \Psi)$ such that $\pi_2 \circ \vartheta = \eta \circ \pi_1$ and $pr_2 \circ \vartheta = \text{id} : G \rightarrow G$, where $pr_2 : E_y \rightarrow G$ is the projection, $y \in N_1$. If $f \in H_p^t(M, N_1)$, then $\eta \circ f := \eta(f(*)) \in H_p^t(M, N_2)$. If $f(s_{0,q}) = y_0$, then $\eta(f(s_{0,q})) = y_0$, since $y_0 \in N_2$. Since $N_2 \subset N_1$, then $H_p^t(M, N_2) \subset H_p^t(M, N_1)$. The parallel transport structure \mathbf{P} is over the same manifold M .

Put $\eta_*(\langle \mathbf{P}_{\hat{\gamma},u} \rangle_{t,H}) = \langle \mathbf{P}_{\eta \circ \hat{\gamma},u} \rangle_{t,H}$, where $\hat{\gamma} : \hat{M} \rightarrow N_1$. In view of Theorems 2.3 and 2.6 [12] $\eta_*(\langle \mathbf{P}_{\hat{\gamma}_1,u} \vee \mathbf{P}_{\hat{\gamma}_2,u} \rangle_{t,H}) = \eta^*(\langle \mathbf{P}_{\hat{\gamma}_1,u} \rangle_{t,H}) \eta_*(\langle \mathbf{P}_{\hat{\gamma}_2,u} \rangle_{t,H})$, and we can put $\eta_*(q^{-1}) = [\eta_*(q)]^{-1}$, consequently, η_* is the group homomorphism. Moreover, for each $g \in (W^M E; N_2, G, \mathbf{P})_{t,H}$ there exists $q \in (W^M E; N_1, G, \mathbf{P})_{t,H}$ such that $\eta_*(q) = g$, since $\gamma : M \rightarrow N_2$ and $N_2 \subset N_1$ imply $\gamma : M \rightarrow N_1$, while the structure group G is the same, hence η_* is the epimorphism.

18. Definition. Let G be a topological group satisfying Conditions 15(A1, A2, C1, C2) such that G is a multiplicative group of the ring \hat{G} , where $1 \leq r \leq 2$. Then define the smashed product G^s such that it is a multiplicative group of the ring $\hat{G}^s := \hat{G} \otimes_l \hat{G}$, where $l = i_{2^r}$ denotes the doubling generator, the multiplication in $\hat{G} \otimes_l \hat{G}$ is

(1) $(a + bl)(c + vl) = (ac - v^*b) + (va + bc^*)l$ for each $a, b, c, v \in \hat{G}$, where $v^* = \text{conj}(v)$.

A smashed product $M_1 \otimes_l M_2$ of manifolds M_1, M_2 over \mathcal{A}_r with $\dim(M_1) =$

$\dim(M_2)$ is defined to be an \mathcal{A}_{r+1} manifold with local coordinates $z = (x, yl)$, where x in M_1 and y in M_2 are local coordinates.

Its existence and detailed description are demonstrated below.

19. Proposition. *The ring \hat{G}^s has a multiplicative group G^s containing all $a + bl \neq 0$ with $a, b \in \hat{G}$. If \hat{G} is a topological or H_p^t differentiable ring over \mathcal{A}_r for $t \geq \dim(G) + 1$, then \hat{G}^s is a topological or H_p^t differentiable over \mathcal{A}_{r+1} ring.*

Proof. For each $1 \leq r \leq 2$ the group G is associative, since the generators $\{i_0, \dots, i_{2^r-1}\}$ form the associative group, when $r \leq 2$. An element $a + bl \in \hat{G}^s$ is non-zero if and only if $(a + bl)(a + bl)^* = aa^* + bb^* \neq 0$ due to 15(A1, A2, C1, C2) and 18(1). For $a + bl \neq 0$ put $u = (a^* - lb^*)/(aa^* + bb^*)$, where $aa^* + bb^* \in G_0$, hence $u(a + bl) = (a + bl)u = 1 \in G_0$, since G_j is commutative for each $j = 0, \dots, 2^r - 1$, where G_j denotes the multiplicative group of the ring \hat{G}_j . For $r \leq 2$ the family of generators $\{i_0, \dots, i_{2^{r+1}-1}\}$ forms the alternative group, hence $\hat{G}^s = \hat{G}_0 i_0 \oplus \dots \oplus \hat{G}_{2^{r+1}-1}$ is alternative, where \hat{G}_j are isomorphic with \hat{G}_0 for each j .

If an addition in \hat{G} is continuous, then evidently $(a + bl) + (c + ql) = (a + c) + (b + q)l$ is continuous. If the multiplication in \hat{G} is continuous, then Formula 18(1) shows that the multiplication in \hat{G}^s is continuous as well.

We have the decomposition $\mathcal{A}_{r+1} = \mathcal{A}_r \oplus \mathcal{A}_r l$. If \hat{G} is H_p^t differentiable, then from the definition of plots it follows, that \hat{G}^s is H_p^t differentiable over \mathcal{A}_{r+1} (see also in details 20(1 – 5)).

20. Theorem. *Let M_1, M_2 and N_1, N_2 be H_p^t manifolds over \mathcal{A}_r with $1 \leq r \leq 2$, and let G be a group satisfying Conditions 15(A1, A2, C1, C2), let also $M_1 \otimes_l M_2, N_1 \otimes_l N_2$ be smashed products of manifolds and G^s be a smashed product group (see Proposition 19), where $\dim(M_1) = \dim(M_2), \dim(N_1) = \dim(N_2), t \geq \max(\dim(M_1), \dim(N_1), \dim(G)) + 1$. Then the wrap group $(W^{M_1 \otimes_l M_2; \{s_0, j, 1 \otimes_l s_0, v, 2: j=1, \dots, k_1; v=1, \dots, k_2\}} E; N_1 \otimes_l N_2, G^s, \mathbf{P}^s)_{t, H}$ is twisted over $\{i_0, \dots, i_{2^{r+1}-1}\}$ and is isomorphic with the smashed product*

$$W^{M_2; \{s_0, v, 2: v=1, \dots, k_2\}} E; N_1, (W^{M_1; \{s_0, j, 1: j=1, \dots, k_1\}} E; N_1, G, \mathbf{P}_1)_{t, H}, \mathbf{P}_2)_{t, H} \otimes_l$$

$$W^{M_2; \{s_0, v, 2: v=1, \dots, k_2\}} E; N_2, (W^{M_1; \{s_0, j, 1: j=1, \dots, k_1\}} E; N_2, G, \mathbf{P}_1)_{t, H}, \mathbf{P}_2)_{t, H} \text{ of twice}$$

iterated wrap groups twisted over $\{i_0, \dots, i_{2^r-1}\}$.

Proof. Let M_b and N_b be H_p^t manifolds over \mathcal{A}_r with $1 \leq r \leq 2, b = 1, 2$ and let G be a group satisfying Conditions 15(A1, A2, C1, C2) such that $E(N_b, G, \pi, \Psi)$ is a principal G -bundle. Consider the smashed products $M_1 \otimes_l M_2, N_1 \otimes_l N_2$ of manifolds and the smashed product group G^s (see Proposition 19), where $t \geq \max(\dim(M_1), \dim(N_1), \dim(G)) + 1$, where $\dim(M_b)$ is a covering dimension of M_b (see [4]), $\dim(M_1) = \dim(M_2), \dim(N_1) = \dim(N_2)$. For $At(M_b) = \{(U_{j,b}, \phi_{j,b}) : j\}$ an atlas of M_b its connecting mappings $\phi_{j,b} \circ \phi_{k,b}^{-1}$ are H_p^t functions over \mathcal{A}_r for $U_{j,b} \cap U_{k,b} \neq \emptyset$,

where $\phi_{j,b} : U_{j,b} \rightarrow \mathcal{A}_r$ are homeomorphisms of $U_{j,b}$ onto $\phi_{j,b}(U_{j,b})$. Then $M_1 \otimes_l M_2$ consists of all points (x, yl) with $x \in M_1$ and $y \in M_2$, with the atlas $At(M_1 \otimes_l M_2) = \{(U_{j,1} \otimes_l U_{q,2}, \phi_{j,1} \otimes_l \phi_{q,2}) : j, q\}$ such that $\phi_{j,1} \otimes_l \phi_{q,2} : U_{j,1} \otimes_l U_{q,2} \rightarrow \mathcal{A}_{r+1}^m$, where m is a dimension of M_1 over \mathcal{A}_r . Express for $z = x + yl \in \mathcal{A}_r$ with $x, y \in \mathcal{A}_r$ numbers x, y in the z representation, then denote by $\theta_{j,q}$ mappings corresponding to $\phi_{j,1} \otimes_l \phi_{q,2}$ in the z representation, hence the transition mappings $\theta_{j,q} \circ \theta_{k,n}^{-1}$ are H_p^t over \mathcal{A}_{r+1} , when $(U_{j,1} \otimes_l U_{q,2}) \cap (U_{k,1} \otimes_l U_{n,2}) \neq \emptyset$. Therefore, $M_1 \otimes_l M_2$ and $N_1 \otimes_l N_2$ are H_p^t manifolds over \mathcal{A}_{r+1} .

In view of the Sobolev embedding theorem each H^t mapping on $M_1 \otimes_l M_2$ or $N_1 \otimes_l N_2$ or G^s is continuous for t satisfying the inequality

$$t \geq \max(\dim(M_1), \dim(N_1), \dim(G)) + 1, \text{ where } \dim(M_1) = \dim(M_2), \dim(N_1) = \dim(N_2).$$

Each locally analytic function $f(x, y) = f_1(x, y) + f_2(x, y)l$ by $x \in U$ and $y \in V$ can be written as the locally analytic function by $z = x + yl$ with values in \mathcal{A}_{r+1} , where U and V are open in \mathcal{A}_r^m , $f_b(x, y)$ is a locally analytic function with values in \mathcal{A}_r^w , $b = 1, 2$, $m, w \in \mathbf{N}$. Indeed, write each variable x_j and y_j through z_j with the help of generators of \mathcal{A}_{r+1} , where $x_j, y_j \in \mathcal{A}_r$, $z_j \in \mathcal{A}_{r+1}$, $x = (x_1, \dots, x_m) \in \mathcal{A}_r^m$, $z = (z_1, \dots, z_m) \in \mathcal{A}_{r+1}^m$ (see Formulas 2.8(2) and Theorem 2.16 [18]). If $z \in \mathcal{A}_{r+1}$, then

- (1) $z = v_0 i_0 + \dots + v_{2^{r+1}-1} i_{2^{r+1}-1}$, where $v_j \in \mathbf{R}$ for each $j = 0, \dots, 2^{r+1}-1$,
- (2) $v_0 = (z + (2^{r+1} - 2)^{-1} \{-z + \sum_{j=1}^{2^{r+1}-1} i_j(z i_j^*)\})/2$,
- (3) $v_s = (i_s(2^{r+1} - 2)^{-1} \{-z + \sum_{j=1}^{2^{r+1}-1} i_j(z i_j^*)\} - z i_j)/2$ for each $s = 1, \dots, 2^{r+1}-1$, where $z^* = \tilde{z}$ denotes the conjugated Cayley-Dickson number z . At the same time we have for $z = x + yl$ with $x, y \in \mathcal{A}_r$, that

- (4) $x = v_0 i_0 + \dots + v_{2^r-1} i_{2^r-1}$ and
- (5) $y = (v_{2^r} i_{2^r} + \dots + v_{2^{r+1}-1} i_{2^{r+1}-1})l^*$,

where $l = i_{2^r}$ denotes the doubling generator.

Therefore, $f(x, y)$ becomes \mathcal{A}_{r+1} holomorphic using the corresponding phrases arising canonically from expressions of x_j, y_j through z_j by Formulas (1 – 5). The set of holomorphic functions is dense in H_p^t in accordance with the definition of this space, hence using a Cauchy net we can consider for each $f_1, f_2 \in H_p^t$ over \mathcal{A}_r a representation of a function $f = f_1 + f_2 l$ belonging to H_p^t over \mathcal{A}_{r+1} (see also [18, 15]).

Then $E(N_1 \otimes_l N_2, G^s, \pi^s, \Psi^s)$ is naturally isomorphic with $E(N_1, G, \pi_1, \Psi_1) \otimes_l E(N_2, G, \pi_2, \Psi_2)$, where $\pi^s = \pi_1 \otimes \pi_2 l : E(N_1 \otimes_l N_2, G^s, \pi^s, \Psi^s) \rightarrow N_1 \otimes_l N_2$ is the natural projection.

If $\gamma : M_1 \otimes_l M_2 \rightarrow N_1 \otimes_l N_2$ is an H_p^t mapping, then $\gamma(z) = \gamma_1(x, y) \times \gamma_2(x, y)l$, where $x \in M_1$ and $y \in M_2$, $z = (x, yl) \in M_1 \otimes_l M_2$, $\gamma_b : M_1 \otimes_l M_2 \rightarrow N_b$. We can write $\gamma_b(x, y)$ as $(\gamma_{b,1}(x))(y)$ a family of functions by x and a

parameter y or as $(\gamma_{b,2}(y))(x)$ a family of functions by y with a parameter x . If $\eta_{b,a} : M_a \rightarrow N_b$, then $\mathbf{P}_{\hat{\eta}_{b,a}, u_b, a}$ denotes the parallel transport structure on M_a over $E(N_b, G, \pi_b, \Psi_b)$.

Then $\mathbf{P}_{\hat{\gamma}, u}^s(z) = [\mathbf{P}_{\hat{\gamma}_{1,1}, u_1; 1}(x)][\mathbf{P}_{\hat{\gamma}_{1,2}, u_1; 2}(y)] \otimes_l [\mathbf{P}_{\hat{\gamma}_{2,1}, u_2; 2}(x)][\mathbf{P}_{\hat{\gamma}_{2,2}, u_2; 2}(y)] \in E_{y_0}(N_1 \otimes_l N_2, G^s, \pi^s, \Psi^s)$ is the parallel transport structure in $M_1 \otimes_l M_2$ induced by that of in M_1 and M_2 , where $u \in E_{y_0}(N_1 \otimes_l N_2, G^s, \pi^s, \Psi^s)$, $u = u_1 \otimes_l u_2$, $u_b \in E_{y_{0,b}}(N_b, G, \pi_b, \Psi_b)$, $y_{0,b} \in N_b$ is a marked point, $b = 1, 2$, $y_0 = y_{0,1} \otimes_l y_{0,2}$. Then \mathbf{P}^s is G^s equivariant. Therefore, $\langle \mathbf{P}_{\hat{\gamma}, u}^s \rangle_{t,H} = \langle \mathbf{P}_{\hat{\gamma}_{1,1}, u_1} \rangle_{t,H} \otimes_l \langle \mathbf{P}_{\hat{\gamma}_{2,2}, u_2} \rangle_{t,H} = \langle [\mathbf{P}_{\hat{\gamma}_{1,1}, u_1; 1}(x)][\mathbf{P}_{\hat{\gamma}_{1,2}, u_1; 2}(y)] \rangle_{t,H} \otimes_l \langle [\mathbf{P}_{\hat{\gamma}_{2,1}, u_2; 2}(x)][\mathbf{P}_{\hat{\gamma}_{2,2}, u_2; 2}(y)] \rangle_{t,H}$, where $\mathbf{P}_{\hat{\gamma}_{b,a}, u_b}$ is the parallel transport structure on $M_1 \otimes_l M_2$ over $E(N_b, G, \pi_b, \Psi_b)$, $b = 1, 2$.

Hence $(W^{M_1 \otimes_l M_2; \{s_{0,j,1} \otimes_l s_{0,v,2}; j=1, \dots, k_1; v=1, \dots, k_2\}} E; N_1 \otimes_l N_2, G^s, \mathbf{P}^s)_{t,H}$ is isomorphic with the smashed product

$$W^{M_2; \{s_{0,v,2}; v=1, \dots, k_2\}} E; N_1, (W^{M_1; \{s_{0,j,1}; j=1, \dots, k_1\}} E; N_1, G, \mathbf{P}_1)_{t,H}, \mathbf{P}_2)_{t,H} \otimes_l W^{M_2; \{s_{0,v,2}; v=1, \dots, k_2\}} E; N_2, (W^{M_1; \{s_{0,j,1}; j=1, \dots, k_1\}} E; N_2, G, \mathbf{P}_1)_{t,H}, \mathbf{P}_2)_{t,H}$$

of iterated wrap groups.

21. Theorem. *There exists a homomorphism of iterated wrap groups $\theta : (W^M E)_{a;\infty,H} \otimes (W^M E)_{b;\infty,H} \rightarrow (W^M E)_{a+b;\infty,H}$ for each $a, b \in \mathbf{N}$, where G is an H_p^∞ group, $E(N, G, \pi, \Psi)$ is the principal H_p^∞ bundle with the structure group G . Moreover, if G is either associative or alternative, then θ is either associative or alternative.*

Proof. Consider iterated wrap groups $(W^M E)_{a;\infty,H}$ as in §4, $a \in \mathbf{N}$. If $\gamma_a : M^a \rightarrow N$, $\gamma_b : M^b \rightarrow N$ are H_p^∞ mappings such that $\gamma_b(s_{0,j_1} \times \dots \times s_{0,j_b}) = y_0$ for each $j_l = 1, \dots, k$ and $l = 1, \dots, b$, then $\gamma := \gamma_a \times \gamma_b : M^a \times M^b \rightarrow N \times N = N^2$, where $M^a \times M^b = M^{a+b}$, $s_{0,j}$ are marked points in M with $j = 1, \dots, k$ and y_0 is a marked point in N , $H_p^\infty = \bigcap_{t \in \mathbf{N}} H_p^t$. This gives the iterated parallel transport structure $\mathbf{P}_{\hat{\gamma}, u; a+b}(x) := \mathbf{P}_{\hat{\gamma}_a, u_a; a}(x_a) \otimes \mathbf{P}_{\hat{\gamma}_b, u_b; b}(x_b)$ on M^{a+b} over $E(N^2, G^2, \pi, \Psi)$, where $u_b \in E_{y_0}(N, G, \pi, \Psi)$, $u = u_a \times u_b \in E_{y_0 \times y_0}(N^2, G^2, \pi, \Psi)$.

The bunch $M^b \vee M^b$ is taken by points s_{j_1, \dots, j_b} in M^b , where $s_{j_1, \dots, j_b} := s_{0,j_1} \times \dots \times s_{0,j_b}$ with $j_1, \dots, j_b \in \{1, \dots, k\}$; $s_{0,j}$ are marked points in M with $j = 1, \dots, k$. Then $(M^a \vee M^a) \times (M^b \vee M^b) \setminus \{s_{j_1, \dots, j_{a+b}} : j_l = 1, \dots, k; l = 1, \dots, a+b\}$ is H_p^t homeomorphic with $M^{a+b} \vee M^{a+b} \setminus \{s_{j_1, \dots, j_{a+b}} : j_l = 1, \dots, k; l = 1, \dots, a+b\}$, since $s_{j_1, \dots, j_a} \times s_{j_{a+1}, \dots, j_{a+b}} = s_{j_1, \dots, j_{a+b}}$ for each j_1, \dots, j_{a+b} . There is the embedding $\text{Diff}_p^\infty(M^a) \times \text{Diff}_p^\infty(M^b) \hookrightarrow \text{Diff}_p^\infty(M^{a+b})$ for each $a, b \in \mathbf{N}$. If $f_a \in \text{Diff}_p^\infty(M^a)$ having a restriction $f_a|_{K_a} = \text{id}$, then $f_a \times f_b \in \text{Diff}_p^\infty(M^{a+b})$ and $f_a \times f_b|_{K_a \times K_b} = \text{id}$ for $K_a \subset M^a$. Put $\theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a}, \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b}) = \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b} \rangle_{\infty, H; a+b}$ is the group homomorphism, where the detailed notation $\langle * \rangle_{t, H; a}$ denotes the equivalence class over the manifold M^a instead of M , $a \in \mathbf{N}$.

Therefore, $\langle \mathbf{P}_{\hat{\gamma} \vee \hat{\eta}, u; a+b} \rangle_{\infty, H; a+b} = \langle \langle \mathbf{P}_{\hat{\gamma}_a \vee \hat{\eta}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b \vee \hat{\eta}_b, u_b; b} \rangle_{\infty, H; b} \rangle_{\infty, H; a+b}$

$$\begin{aligned}
&= \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \langle \mathbf{P}_{\hat{\eta}_a, u_a; a} \rangle_{\infty, H; a} \rangle \otimes \langle \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b} \langle \mathbf{P}_{\hat{\eta}_b, u_b; b} \rangle_{\infty, H; b} \rangle \\
&\rangle_{\infty, H; a+b} \\
&= \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b} \rangle \langle \langle \mathbf{P}_{\hat{\eta}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\eta}_b, u_b; b} \rangle_{\infty, H; b} \rangle \\
&\rangle_{\infty, H; a+b} \\
&= \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b} \rangle_{\infty, H; a+b} \langle \langle \mathbf{P}_{\hat{\eta}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\eta}_b, u_b; b} \rangle_{\infty, H; b} \rangle_{\infty, H; a+b} \\
&= \theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a}, \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b}) \theta(\langle \mathbf{P}_{\hat{\eta}_a, u_a; a} \rangle_{\infty, H; a}, \langle \mathbf{P}_{\hat{\eta}_b, u_b; b} \rangle_{\infty, H; b}).
\end{aligned}$$

Thus θ is the group homomorphism.

The mapping $H_p^\infty(M^a, N) \times H_p^\infty(M^b, N) \ni (\gamma_a \times \gamma_b) \mapsto (\gamma_a, \gamma_b) \in H_p^\infty(M^{a+b}, N^2)$ is of H_p^∞ class. The multiplication in G^v is H_p^∞ for each $v \in \mathbf{N}$, since it is such in G , since the multiplication in G^v is $(a_1, \dots, a_v) \times (b_1, \dots, b_v) = (a_1 b_1, \dots, a_v b_v)$, where G^v is the v times direct product of G , $a_1, \dots, a_v, b_1, \dots, b_v \in G$.

The iterated wrap group $(W^M E)_{l;t,H}$ for the bundle E is the principal G^{kl} bundle over the iterated commutative wrap group $(W^M N)_{l;t,H}$ for the manifold N , since the number of marked points in M^l is kl , where E is the principal G bundle on the manifold N , $l \in \mathbf{N}$. Thus the iterated wrap group is associative or alternative if such is G . In view of Proposition 7 and Remark 4 the homomorphism θ is of H_p^∞ class. From the wrap monoids it has the natural H_p^∞ extension on wrap groups.

If G is associative, then

$$\begin{aligned}
&\langle \mathbf{P}_{\hat{\gamma}, u; a+b+v} \rangle_{\infty, H; a+b+v} = \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b} \rangle_{\infty, H; a+b} \otimes \langle \mathbf{P}_{\hat{\gamma}_v, u_v; v} \rangle_{\infty, H; v} \rangle_{\infty, H; a+b+v} \\
&= \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \rangle \otimes \langle \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b} \otimes \langle \mathbf{P}_{\hat{\gamma}_v, u_v; v} \rangle_{\infty, H; v} \rangle_{\infty, H; a+b+v} = \\
&\theta(\theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{t, H; a}, \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{t, H; b}), \langle \mathbf{P}_{\hat{\gamma}_v, u_v; v} \rangle_{t, H; v}) \\
&\theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{t, H; a}, \theta(\langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{t, H; b}, \langle \mathbf{P}_{\hat{\gamma}_v, u_v; v} \rangle_{t, H; v})),
\end{aligned}$$

consequently, θ is the associative homomorphism.

If G is alternative, then

$$\begin{aligned}
&\langle \mathbf{P}_{\hat{\gamma}, u; a+a+b} \rangle_{\infty, H; a+a+b} = \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \rangle_{\infty, H; a+a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b} \rangle_{\infty, H; a+a+b} \\
&= \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \rangle \otimes \langle \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{\infty, H; a} \otimes \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{\infty, H; b} \rangle_{\infty, H; a+a+b} = \\
&\theta(\theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{t, H; a}, \langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{t, H; a}), \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{t, H; b}) \\
&\theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{t, H; a}, \theta(\langle \mathbf{P}_{\hat{\gamma}_a, u_a; a} \rangle_{t, H; a}, \langle \mathbf{P}_{\hat{\gamma}_b, u_b; b} \rangle_{t, H; b})),
\end{aligned}$$

consequently, the homomorphism θ is alternative from the left, analogously it is alternative from the right.

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